A Complete Characterization of non-Euclidean Lines.

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Abstract. *In this paper we give a complete characterization of non-archimedean lines. We show, through an investigation inside the non-archimedean axiomatic theory, that each of these lines is isomorphic to a particular direct product* $G \times R$ *, where G is an ordered group and R is the ordered field of real numbers. In such a way, also the real line can be regarded as the direct product when* $G = \{0\}$*. Finally, we are able to prove that, eventually, an isomorphism between two non necessarily archimedean lines* $L_1 = G_1 \times R$ *and* $L_2 = G_2 \times R$ *exists if, and only if, the groups* G_1 *and* G_2 *are each other isomorphic.*

1.- Background and new results.

Throughout all the studies, the analisys of foundations and the historical investigations on developments of the Euclidean Geometry we may note that a very small attemption has been given to the idea of the axiomatic euclidean line as a stand alone structure, i.e. not inducted by the axiomatic of the euclidean plane. Howewer, it is well known that an hypotetic set of axioms for the euclidean line is necessarily equivalent to the real numbers structure.

We remind that in this matter M. Pasch [3] and G. Peano [1], [2] wrote fundamental papers. First of all, Peano introduced a set of axioms in order to characterize the euclidean line. The line structure so built was a study of order perspectivity only! Moreover the Peano axioms are equivalent to a structure of a totally ordered set! The Peano- line structure – i.e. a totally ordered set - was incomplete with respect to congruence and under continuity perspective. The situation was left open, so, no conclusion were possible about the possibility of an isomorphism between the euclidean line and the real line. Instead, Pasch studies investigated the euclidean line as a subset of the plane and so its properties are consequent to the plane axioms.

In [10] we presented a list of axioms, which contains the Peano's axioms of order, from which we have showed the isomorphism with the real line. Moreover, in [10] we prove that the Peano's axioms of order are equivalent to the axioms of a total ordered set.

So, in [11] we present a new list of axioms, precisely starting from a total ordered set, and we construct, also in this case, an isomorphism with the real line.

Referring to [10], [11], an eucliden real line is an algebraic structure $(L, \leq, \equiv)$, where L is a non-empty set, "$\leq$" is a total order on L and "$\equiv$" is an equivalence relation on the set of segments of poset $(L, \leq)$, called "congruence". Let us assume that the following axioms holds.

A1.- On L there exists at least two distinct points.

A2.- (Order existence) On L is given a total order relation $\leq$. Througout this relation are introduced the notions of open and closed segment:

$$(A, B) = \{X \in L : A < X < B\}, \ [A, B] = \{X \in L : A \leq X \leq B\}, \ \forall A, B \in L.$$

A3.- (Extensibility) If $A, C \in L$ are any two distinct points, it exists at least one point $D \in L$ such that $C \in (A, D)$.

A4.- (Density) If $A, C \in L$ are any two distinct points, it exists at least one point $B \in L$ such that $B \in (A, C)$.

A5.- (Congruence existence) Let S be the set of all possible segments on L. On S there exists an equivalence relation "≡□, called *congruence*.

A6.- (Segment reversibility) Every segment (A, B) is congruous to his opposite $-(A, B)$, where the opposite of a segment is the segment containing the same points, but ordered in the opposite direction.

A7.- (Addability) Let (A, B), (B, C) be any two disjoint segments, so as (A', B') and (B', C'). If $(A, B) \equiv (A', B')$ and $(B, C) \equiv (B', C')$ then it follows $(A, C) \equiv (A', C')$.

A8.- (Transport) Let it given the points A, B, A' and a semiline with origin A' (i.e. the set of all points starting from A' in one direction: in this way there are two semilines with A' as origin point). Than there exists exactly one point C on this semiline such that $(A, B) \equiv (A', C)$.

A9.- (Eudosso-Archimede's principle) Let (A, B), (C, D) be any two segments in S. Than there exists a natural number n such that by adding n times the segment (A, B) from the point C through the semiline containing D, the ending point B' of the segment so obtained satisfies $D < B'$.

A10.- (Cantor's principle) Any couple S_1, S_2 of contiguous classes on L has got a separation element.

In [10], [11] we have proved the following result: "Let $(L, \leq, \equiv)$ be a structure satisfying the axioms A1, …, A10, defined above. Then $(L, \leq, \equiv)$ is isomorphic to the structure $(R, \leq, =)$, where R is the usual field of real numbers".

In this paper we will call *non-archimedean line* any structure $(L, \leq, \equiv)$ which satisfies the axioms A1, …, A10, but, eventually, doesn't satisfy axiom A9.

A new general model of non-archimedean line can be given by generalizing the well known Veronese's model (see [13], [14] and [16]), which was purely geometric and embedded in an euclidean plane. This algebraic model, if applied to Veronese's idea too, allow us to give a numerical representation of such famous model, without regards to the euclidean plane "box".

Let $(G, +, \leq)$ an ordered abelian group and let $(R, +, \leq)$ the additive group of the reals. Let's define, as well, an order relation $\leq$ and an equivalence relation $\cong$ on the cartesian product $G \times R$, in the following intuitive manner:

– $\forall g_1, g_2 \in G, \forall r_1, r_2 \in R,$ (1)

$(g_1, r_1) \leq (g_2, r_2)$ (on $G \times R$) $\Leftrightarrow$ $g_1 < g_2$ (on G), or $g_1 = g_2$ and $r_1 \leq r_2$ (on R);

– $\forall g_1, g_2, g_3, g_4 \in G, \forall r_1, r_2, r_3, r_4 \in R,$ (2)

$[(g_1, r_1), (g_2, r_2)] \cong [(g_3, r_3), (g_4, r_4)]$ (on $G \times R$) $\Leftrightarrow$ $g_2 - g_1 = g_4 - g_3$ (on G), $r_2 - r_1 = r_4 - r_3$ (on R).

It can be easily shown that the line $(G \times R, \leq, \cong)$ could be both archimedean or non-archimedean. In particular, $(G \times R, \leq, \cong)$ is archimedean if, and only if, G is reduced to the neutral element "0_G". Moreover, if G coincides with the ordered additive group Z of relative integers, then the model $(Z \times R, \leq, \cong)$ is isomorphic to Veronese's one, as proved in [9].

The theorems given below may clear completely the question of non-archimedean characterization.

***Theorem* 1**. Let $(L, \leq, \equiv)$ be a line which satisfies the axioms A1, …, A8, A10 but, eventually, doesn't satisfy the axiom A9, or the Eudosso-Archimede's principle. Then it exists an ordered abelian group G such that $(L, \leq, \equiv)$ is isomorphic to the structure $(G \times R, \leq, \cong)$.

***Theorem* 2**. Let $(G_1 \times R, \leq, \cong)$ and $(G_2 \times R, \leq, \cong)$ be any two lines not necessarily archimedean. These lines are isomorphic if, and only if, also G_1 and G_2 are each other isomorphic.

The proofs of theorems 1 and 2 are described in the next sections. In the first proof, due to his length, we have considered a series of steps which may clear more the underlying ideas. Such proof, however, has been given in a different manner also in [12], in which we report the axiomatic construction of the group G. The proof of the second theorem appears also in [12], but just between the lines.

2.- Proof of Theorem 1.

Step 1. Definitions of sub-alignment and sum over L.

Let's first define the key relation of *sub-alignment* over L through the classic notion of commensurability.

Definition. Let A, B be two point over L; A is sub-aligned with B, and we write $A \sim B$, if each segment (H, K) contained in the segment (A, B) is commensurable with (A, B); in other words it exists a natural n such that by adding n times the segment (H, K) from the point A through the semiline containing B, the ending point B' of the segment so obtained satisfies $B < B'$ (we may write $n(H, K) > (A, B)$, as well).

Let's note that, by doing so, we are going to consider the local validity of the archimedean principle. In fact, by considering the L-points which are sub-aligned with A, we consider the archimedean principle as a local property around A and take on the points which may be overtaken by any segment contained in (A, B). In this manner, we could say that we consider a set of L-points which are "archimedean" with A. Besides, by definition, if the property doesn't hold for the points A and B, we say that A is "not sub-aligned" with B, and write A !~ B.

Moreover, as we may easily prove, "~"is an equivalence relation.[1]

We can also define a binary operation of *sum* over L through the notion of congruence and by means of the transport axiom A8.

Definition. Let $O \in L$ be a fixed point. The operation "+": $L \times L \to L$ is so defined: (15)

$A + B = C$ where C is the following point

a) if $O \leq A$, C is the point such that $B \leq C$ and $(O, A) \equiv (B, C)$;

b) if $A \leq O$, C is the point such that $C \leq B$ and $(A, O) \equiv (C, B)$.

We can easily see that the structure $(L, +)$ is an abelian group with neutral element O and with opposite element of any point A given by the point $–A$ such that $(–A, O) \equiv (O, A)$ (see [10]).

We can easily prove that $(A, B) \equiv (C, D)$ if, and only if $B + (–A) = D + (–C)$. In fact, let K be the point such that $(O, K) \equiv (A, B) \equiv (C, D)$ then, by the sum definition on L, it holds $K + A = B$ and $K + C = D$, or $B + (–A) = K = D + (–C)$.

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[1] In particular, the transitivity property holds. Let $A \sim B$, $B \sim C$ and $(H, K) \subset (A, B)$. Then it exist n, m naturals such that $n(H, K) > (A, B)$ and, if K' is such that $(H, K) \equiv (B, K')$, $m(B, K') > (B, C)$. Then it is clear that $(n+m)(H, K) > (A, C)$ holds, or $A \sim C$.

Step 2. Isomorphism of the classes L_j with the real numbers.

Through the sub-alignment relation we create a partition over *L*, into a series of classes L_j called *sub-line* which is crucial in the following.

Proposition. Let $T = L/_{\sim} = \{L_j\}_{j \in J}$ be the quotient set, that we call *trasversal quotient*, inducted over *L* by the sub-alignment relation. Then each structure $(L_j, \leq, +)$ is isomorphic to the structure $(R, \leq, +)$, where *R* is the set of real numbers.

Proof. Each sub-line L_j satisfy all the axioms A1, ..., A10. In fact:

- on each L_j there are at least two distinct points (A1 holds). This fact follows straight if A10 holds over *L*, since $L_j = L$. Otherwise, if A10 doesn't hold over *L*, it exists at least one sub-line L_α containing at least two points. If it weren't so, in each segment, each sub-segment would be commensurable with it, contradicting the fact that A10 doesn't hold. Then, by means of the Transport axiom over *L*, each class L_j has got at least two points, since two sub-aligned points cannot form a segment which is congruent to a segment whose extremal points are not sub-aligned (see [12], prop. 3.5);
- the existence of order and congruence relations on each L_j, so as segment reversibility and addability, follows straight (A2, A5, A6 and A7 hold);
- let *A*, *C* be distinct points on L_j. It exists, by means of the Transport axiom over *L*, a point *D* on *L* such that $(A, C) \equiv (C, D)$. Being $A \sim C$, it holds also $C \sim D$, then $D \in L_j$ (A3 holds);
- let $A, C \in L_j \subset L$. Than it exists one point $B \in L$ such that $B \in (A, C)$, but $A \sim C$ implies $A \sim B$, or $B \in L_j$ (A4 holds);
- let $A, B, A' \in L_j$, and suppose a semiline with origin *A'* is given. Than it exists a unique point *C* on this semiline such that $(A, B) \equiv (A', C)$ and, from $A \sim B$, it follows $A' \sim C$, i.e. $C \in L_j$ (A8 holds);
- Su ogni classe d'equivalenza L_j vale l'assioma di Eudosso-Archimede.
- let $A, B, C, D \in L_j$. Then $A \sim B$ and $C \sim D$. Let $H \in L_j$ be the point obtained by transporting (*A*, *B*) over (*C*, *D*), starting form *C* towards *D*, in such a way that $(A, B) \equiv (C, H)$. If $D < H$ then $1(A, B) > (C, D)$; otherwise, if $H < D$, being $C \sim D$, it exists a natural *n* such that $n(A, B) > (C, D)$. Then Eudosso-Archimede's principle holds;
- any couple of contiguous classes S_1, S_2 over any $L_j \subset L$ admits, by means of the Cantor's principle over *L*, a separation element on *L*. If $A \in S_1$, $B \in S_2$ and *H* is the separation element of the classes S_1, S_2, since $A \leq H \leq B$, from $A \sim B$ it follows $A \sim H$, i.e. $H \in L_j$ (A10 holds).

At this stage, the validity of all the axioms A1,..., A10 over (L_j, ≤, +) implies an isomorphism between (L_j, ≤, +) and (R, ≤, +). This fact has been proved in [11] (see, in particular, theorem 4.11).

Step 3. The line L can be both archimedean or non-archimedean.

In the previous step we have introduced the trasversal quotient T. Now we are in position to prove that the validity of the archimedean principle over the line L is connected to the cardinality of T which can be infinite or equal to 1.

Proposition. Eudosso-Archimede's principle holds over L if, and only if, the cardinality of T is finite (and equal to 1).

Proof. If A9 holds over L, each point A is sub-aligned with each other point, so T contains just one equivalence class L_j.

If A9 doesn't hold over L, it exist at least two distinct sub-lines $L_\alpha, L_\beta \in$ T. Let $A \in L_\alpha$, $B \in L_\beta$ and suppose $A < B$ (in case $B < A$ the proof is analogous). The points A, B are not sub-aligned and the transport of the segment (A, B) from B towards the direction opposite to A, reach a point K not sub-aligned with B such that $(A, B) \equiv (B, K)$. Since $A < B < K$ with $K \notin L_\beta$, it follows that $K \in L - (L_\alpha \cup L_\beta)$ because, otherwise, if it were $K \in L_\alpha$, it would hold $(A, B) \subset (A, K)$ and A ~ B, contradicting the hypothesis. Then it exists another sub-line $L_\gamma \subset L - (L_\alpha \cup L_\beta)$. By repeating the reasoning, or transporting infinite many times the segment (A, B), it remains proved the infiniteness of the number of sub-lines L_j, which completes the thesis.

Step 4. Definitions of order, congruence and sum over T.

In this step we introduce two relations and a binary operation on T: the order "≤", the congruence "≅",and the sum over T. These new definitions come out from the order on L,

Definition. On T is inducted an order relation "≤" defined as follows. Let $L_\alpha, L_\beta \in T$, then

i) $L_\alpha < L_\beta$ if, for any $A \in L_\alpha$, $B \in L_\beta$, it holds $L_\alpha \neq L_\beta$ and $A < B$;

ii) $L_\alpha \leq L_\beta$ if $L_\alpha < L_\beta$ or $L_\alpha = L_\beta$.

As we may easily prove, the order just introduced:

- is well-defined, i.e. the definition works independently from particular representatives chosen on the classes;

- creates a structure $(T, \leq)$ which is totally ordered. The proof is a trivial application of total order on L (see [12] prop. 3.13, for more details).

If we use the notation L_A to indicate the sub-line which contains the point A, the strict connection between the orders on L and on T is explained by the following properties:

- if $A < B$ then $L_A \leq L_B$; (13)
- if $L_A < L_B$ then $A < B$.

Througout this relation we can introduce also the notions of *open* and *closed trasversal segment*:

$$(L_\alpha, L_\beta) = \{ L_j : L_\alpha < L_j < L_\beta \} \quad [L_\alpha, L_\beta] = \{ L_j : L_\alpha \leq L_j \leq L_\beta \} \quad \forall L_\alpha, L_\beta \in T .$$

In particular we'll consider the set $S_T = \{(L_\alpha, L_\beta) : L_\alpha, L_\beta \in T\}$ of open trasversal segments over T.

In this new set we can introduce a congruence relation defined as follows.

Definition. A segment (L_α, L_β) is *traversally congruent* to a segment (L_γ, L_δ) if, chosen any points $A \in L_\alpha$, $B \in L_\beta$, $C \in L_\gamma$, $D \in L_\delta$, the transport over L of the segment (A, B) from the point C towards the point D, reaches a point K such that $(A, B) \equiv (C, K)$ and $(L_C, L_K) = (L_\gamma, L_\delta)$. If (L_α, L_β) is *traversally congruent* to a segment (L_γ, L_δ), we write $(L_\alpha, L_\beta) \cong (L_\gamma, L_\delta)$.

Note that the following properties hold:

- if $(A, B) \equiv (C, D)$ then $(L_A, L_B) \cong (L_C, L_D)$. Let $K \in L$ be such that $(C, K) \equiv (A, B)$ then, from the transport axiom over L, and for the uniqueness of the point, it holds $K = D$. It follows $(L_A, L_B) \cong (L_C, L_K) = (L_C, L_D)$;
- "$\cong$" is well defined. Let $A, A' \in L_A$, $B, B' \in L_B$, $C, C' \in L_C$, $D, D' \in L_D$ with $(L_A, L_B) \cong (L_C, L_D)$. Then, obviously, $(L_{A'}, L_{B'}) = (L_A, L_B) \cong (L_C, L_D) = (L_{C'}, L_{D'})$;
- "$\cong$" is an equivalence relation. Let $(L_A, L_B) \cong (L_C, L_D)$ and $(L_C, L_D) \cong (L_E, L_F)$. Suppose $(A, B) \equiv (C, K) \equiv (E, H)$, then $(L_C, L_D) \cong (L_A, L_B) \cong (L_C, L_K)$, that is $L_D = L_K$, and $(L_E, L_F) \cong (L_C, L_D) = (L_C, L_K) \cong (L_E, L_H)$, that is $L_F = L_H$. Then $(L_A, L_B) \cong (L_E, L_H) = (L_E, L_F)$ and transitivity holds. Riflessivity and simmetry, instead, follows straight from definitions;

Now we are in position to define a binary sum operation over T.

Definition. The sum operation "$+_L$" over L induct a sum operation "+": $T \times T \to T$ on the set of sub-lines (which, we recall, are equivalence classes). This operation is defined setting $L_\alpha + L_\beta = L_\gamma$ if, for any points $A \in L_\alpha$, $B \in L_\beta$, it holds $A +_L B = C$ with $C \in L_\gamma$.

We can easily prove that the sum just introduced:

- is well defined. Let $A, A' \in L_\alpha$, $B, B' \in L_\beta$, such that $A + B = C$, with $C \in L_\gamma$, and $A' + B' = C'$. Suppose that $A \leq A'$, $B \leq B'$ **(...)**. Then it exists $H \in L$ such that $(A, A') \equiv (C, H)$ which, being $A \sim A'$, implies $C \sim H$. By $(O, A) \equiv (B, C)$ and $(A, A') \equiv (C, H)$ it follows $(O, A') \equiv (B, H)$ and by means of tne sum over L, from $(O, A') \equiv (B', C')$ it follows $(B', C') \equiv (B, H)$. Since $B \sim B'$, it holds also $C' \sim H$, or $C' \in L_\gamma$.
- creates a structure $(T, +)$ which is an abelian group. This property is naturally deducted by definition of sum over T and by the fact that $(L, +)$ is an abelian group. For example, L_O is the neutral sub-line and the opposite element of any sub-line L_A is the sub-line L_{-A} .

We have proved, finally, that $(T, +, \leq)$ is a totally ordered abelian group.
To conclude this step, we remark that the axioms defined on L induct similar properties onto T. The properties which make exception are:

- T may contain just one element, as proved in step 3;
- if $L_\alpha, L_\beta \in T$ are any two distinct sub-lines, it may not exist any sub-line $L_\gamma \in T$ such that $L_\gamma \in (L_\alpha, L_\beta)$. In other words, on T may not hold the *density* property;
- we cannot say anything about the validity of Eudosso-Archimede's principle on T, i.e. it may hold or not;
- similarly, we cannot say anything about the validity of Cantor's principle on T, i.e. it may hold or not.

Step 5. Isomorphism between L and G × R.
In this step we finally show that order and congruence create a structure over L that is isomorphic to the general model $G \times R$ introduced in (1), (2).

Definition. Let O be the point already fixed in the sum definition (15) over L, and let $U \in L_O$ be a new fixed point. On each sub-line L_j we consider two points O_j, U_j such that the set $\{O_j, U_j\}_{j \in J}$ satisfies:

i) $O_\alpha + O_\beta = O_\gamma$ if $L_\alpha + L_\beta = L_\gamma$; (???)

ii) $(O, U) \equiv (O_j, U_j)$ for any $j \in J$.

The aim of the last definition is to fix a referrement point O_j on each sub-line L_j and a segment (O_j, U_j) which give us a standard unity measure on the whole set of sub-lines.

Some consequences can be deducted:

- if $A\in L_\alpha$, $B\in L_\beta$ e $C\in L_\gamma$ are such that $A + B = C$ and $A_O, B_O, C_O \in L_O$ are such that $(O, A_O) \equiv (O_\alpha, A)$, $(O, B_O) \equiv (O_\beta, B)$ and $(O, C_O) \equiv (O_\gamma, C)$, then $A_O + B_O = C_O$. We know, by the construction of the previous points O_j, that $O_\alpha + O_\beta = O_\gamma$, and from the hypothesys it hold $A_O + O_\alpha = A$, $B_O + O_\beta = B$, $C_O + O_\gamma = C$. Then, being $A + B = C$, it follows that $A_O + O_\alpha + B_O + O_\beta = C_O + O_\gamma$, i.e. $A_O + B_O = C_O$ by L group properties.
- if $A\in L_\alpha$, $B\in L_\beta$, $C\in L_\gamma$, $D\in L_\delta$ are such that $(A, B) \equiv (C, D)$ and if $A_O, B_O, C_O, D_O \in L_O$ are such that $(O, A_O)\equiv(O_\alpha, A)$, $(O, B_O)\equiv(O_\beta, B)$, $(O, C_O)\equiv(O_\gamma, C)$, $(O, D_O)\equiv(O_\delta, D)$, then $(A_O, B_O)\equiv(C_O, D_O)$. The proof derives from the group properties on L and by the equations $A + D = B + C$, $A_O + O_\alpha = A$, $B_O + O_\beta = B$, $C_O + O_\gamma = C$, $D_O + O_\delta = D$ and $O_\alpha + O_\delta = O_\beta + O_\gamma$ which imply $A_O + D_O = B_O + C_O$, i.e. $(A_O, B_O) \equiv (C_O, D_O)$.

Now we can proove the following result, which may clear the relationship between the congruences on L, T and each L_j.

Proposition. $(A, B) \equiv (C, D)$ if, and only if, $(L_A, L_B) \cong (L_C, L_D)$ and $(A_O, B_O) \equiv (C_O, D_O)$.

Dim. The implication "⇒" has been proved in the last two steps of this section. Let's suppose, now, that $(L_A, L_B) \cong (L_C, L_D)$ and $(A_O, B_O) \equiv (C_O, D_O)$. First of all, it hold $O_B + (-O_A) = O_D + (-O_C)$ by the congruence of sub-lines and $B_O + (-A_O) = D_O + (-C_O)$ by the congruence of the corrispondent points on L_O. Then $O_B + (-O_A) + B_O + (-A_O) = O_D + (-O_C) + D_O + (-C_O)$ and, since $O_A + A_O = A$, $O_B + B_O = B$, $O_C + C_O = C$, $O_D + D_O = D$, it follows $B + C = O_B + B_O + O_C + C_O = O_A + A_O + O_D + D_O = A + D$, i.e. $(A, B) \equiv (C, D)$.

At this point we can introduce the general isomorphism ϕ, defined by means of the canonical projection from L onto the trasversal quotient T and the isomorphisms between each sub-line L_j and the real line.

Definition. First of all we define the canonical projection $\eta : L \to T$, which for any $A \in L$ is such that

i) $\eta(A) = L_A$.

We consider also, for any sub-line $L_j \in T$, the applications $\psi_j : L_j \to R$ such that:

ii) $\psi_j(O_j) = \psi_O(O) = 0$;
iii) $\psi_j(U_j) = \psi_O(U) = 1$;
iv) $\psi_j(rU_j) = \psi_O(rU) = r$;
where $\psi_O : L_O \rightarrow R$, and a generalized application $\psi : L \rightarrow R$ such that
v) $\psi(A) = \psi_{\eta(A)}(A)$ for any $A \in L$.
Then we define an application $\phi : L \rightarrow T \times R$, such that for any point $A \in L$ gives back a couple $(t, a) \in T \times R$ which is so obtained:
vi) $\phi(A) = (\eta(A), \psi(A))$. (???)

Note that each application ψ_j is an isomorphism between the structures $(L_j, \leq, \equiv)$ and $(R, \leq, \equiv)$. This fact has been proved in [11].
From this fact, by definition,
- if A_O is such that $(O, A_O) \equiv (O_{\eta(A)}, A)$, it holds $\psi(A)= \psi_{\eta(A)}(A) = \psi_O(A_O) = \psi(A_O)$, or $\psi(O_{\eta(A)} + A_O) = \psi(A_O)$;
- for any $A, B \in L$, it holds $\psi(A + B) = \psi(A) + \psi(B)$.
  Indeed, if $A + B = C$, $A \equiv O_{\eta(A)} + A_O$ and $B \equiv O_{\eta(B)} + B_O$, we have
  $\psi(A + B) = \psi(O_{\eta(A)} + A_O + O_{\eta(B)} + B_O) = \psi(O_{\eta(C)} + A_O + B_O) = \psi(A_O + B_O) = \psi_O(A_O + B_O)$
  $= \psi_O(A_O) + \psi_O(B_O) = \psi(A_O) + \psi(B_O) = \psi(A) + \psi(B)$.

Definition. On the cartesian product $T \times R$ we can define, as in (1) and (2), an order relation "$\leq$" and an equivalence relation "$\cong$" on the set of segments. For any L_1, L_2, L_3, $L_4 \in T$, for any $r_1, r_2, r_3, r_4 \in R$,
- $(L_1, r_1) \leq (L_2, r_2) \Leftrightarrow L_1 < L_2$ (on G), or $L_1 = L_2$ and $r_1 \leq r_2$ (on R); (16)
- $[(L_1, r_1), (L_2, r_2)] \cong [(L_3, r_3), (L_4, r_4)] \Leftrightarrow L_2 - L_1 = L_4 - L_3$ (on T), $r_2 - r_1 = r_4 - r_3$ (on R).

This definition show in a very clear way that $T \times R$ can be regarded as a copy of the model $G \times R$ defined in the first section. So we can definitively show the existence of an isomorphism between L and $G \times R$.

Proposition. The application ϕ is an isomorphism between $(L, \leq, \equiv)$ and $(T \times R, \leq, \cong)$
Proof. The application ϕ is a one-to-one mapping. If $A, B \in L$ are such that $\phi(A) = \phi(B)$, by definition it follows $\eta(A) = \eta(B)$ and $\psi_{\eta(A)}(A) = \psi(A) = \psi(B) = \psi_{\eta(B)}(B)$, which mean that A and B are on the same sub-line and, under the isomorphism $\psi_{\eta(A)}$, have the same correspondent in R, i.e. $A = B$.

The order is preserved. In fact, $A \leq B$ on L implies $\eta(A) = L_A < L_B = \eta(B)$, or $L_A = L_B$ and $r_A = \psi(A) = \psi_{\eta(A)}(A) \leq \psi_{\eta(A)}(B) = r_B$. In any case it holds $\phi(A) = (L_A, r_A) \leq (L_B, r_B) = \phi(B)$.
Moreover, the congruence is preserved. Let $(A, B) \equiv (C, D)$ on L. This implies $(L_A, L_B) \cong (L_C, L_D)$ on T, i.e. $L_B - L_A = L_D - L_C$, and $(A_O, B_O) \equiv (C_O, D_O)$ on L_O , i.e. $B_O - A_O = D_O - C_O$, where A_O, B_O, C_O, D_O are such that $(O, A_O) \equiv (O_{\eta(A)}, A)$, $(O, B_O) \equiv (O_{\eta(B)}, B)$, $(O, C_O) \equiv (O_{\eta(C)}, C)$, $(O, D_O) \equiv (O_{\eta(D)}, D)$. Since the applications ψ_O is an isomorphism, it holds $\psi(B) - \psi(A) = \psi_O(B_\mathrm{O}) - \psi_O(A_\mathrm{O}) = \psi_O(D_\mathrm{O}) - \psi_O(C_\mathrm{O}) = \psi(D) - \psi(C)$. Combining these relations it holds $(\phi(A), \phi(B)) = ((L_A, \psi(A)), (L_B, \psi(B))) \cong ((L_C, \psi(C)), (L_D, \psi(D))) = (\phi(C), \phi(D))$, i.e. the thesis.

3.- Proof of Theorem 2.

3.- Proof of Theorem 2.

Let $(G, +, \leq)$, $(G', +, \leq)$ be any ordered abelian groups, $(R, +, \leq)$ be the ordered additive group of the reals, and $(G \times R, \leq, \cong)$, $(G' \times R, \leq, \cong)$ be the lines whose relations are defined in (1), (2). Our aim is to prove that:

$$(G \times R, \leq, \cong) \text{ is isomorphic to } (G' \times R, \leq, \cong) \;\Leftrightarrow\; (G, +, \leq) \text{ is isomorphic to } (G', +, \leq).$$

"$\Leftarrow$".
Let $(G, +, \leq)$ and $(G', +, \leq)$ be mutually isomorphic and assume $\delta : G \to G'$ is the isomorphism between such structures, so that δ is a 1-1 onto function, $\delta(g_0) \leq \delta(g_1)$ for any $g_0 \leq g_1$ in G, and $\delta(g_2 + g_3) = \delta(g_2) + \delta(g_3)$ for any $g_2, g_3 \in G$.
We now define the application $f : G \times R \to G' \times R$ such that $f(g_1, r_1) = (\delta(g_1), r_1)$.
This new application is a bijection. In fact:

- if (g', r) is any element in $G' \times R$, there exists an element $g \in G$, because δ is a bijection by definition, such that $\delta(g) = g'$. This implies that $f(g, r) = (\delta(g), r) = (g', r)$;
- if $(g_1, r_1) \neq (g_2, r_2)$, that is, $g_1 \neq g_2$ or $r_1 \neq r_2$, then, should $g_1 \neq g_2$, it follows that $\delta(g_1) \neq \delta(g_2)$. In any case, it holds that $f(g_1, r_1) = (\delta(g_1), r_1) \neq (\delta(g_2), r_2) = f(g_2, r_2)$.

If $(g_0, r_0) \leq (g_1, r_1) \in G \times R$ holds, then, by definition, $g_0 < g_1$ or $g_0 = g_1$ and $r_0 \leq r_1$, that is, $\delta(g_0) < \delta(g_1)$, or $\delta(g_0) = \delta(g_1)$ and $r_0 \leq r_1$. It follows that $(\delta(g_0), r_0) \leq (\delta(g_1), r_1) \in G' \times R$, that is, $f(g_0, r_0) \leq f(g_1, r_1)$ and thus f is an isomorphism between ordered sets.
Similarly, if $[(g_1, r_1), (g_2, r_2)] \cong [(g_3, r_3), (g_4, r_4)]$ holds on $G \times R$ then, by definition, it follows that $g_2 - g_1 = g_4 - g_3$ on G and $r_2 - r_1 = r_4 - r_3$ on R. In particular, from the first equation, we obtain $g_2 + g_3 = g_1 + g_4$. Thus, we have $\delta(g_2) + \delta(g_3) = \delta(g_2 + g_3) = \delta(g_1 + g_4) = \delta(g_1) + \delta(g_4)$, or $\delta(g_2) - \delta(g_1) = \delta(g_4) - \delta(g_3)$, from which it holds that $[(\delta(g_1), r_1), (\delta(g_2), r_2)] \cong [(\delta(g_3), r_3), (\delta(g_4), r_4)]$. This also implies that the congruence is preserved between $G \times R$ and $G' \times R$ under the function f.
In conclusion, we have proven that f is an isomorphism between $(G \times R, \leq , \cong)$ and $(G' \times R, \leq , \cong)$.

"$\Rightarrow$".
Let $(G \times R, \leq , \cong)$ and $(G' \times R, \leq , \cong)$ be each other isomorphic and suppose $f : G \times R \to G' \times R$, with $f(g, r) = (f_1(g, r), f_2(g, r))$, is the isomorphism between such structures so that f is a bijection, $f(g_0, r_0) \leq f(g_1, r_1)$ for any $(g_0, r_0) \leq (g_1, r_1)$ in $G \times R$, and $[f(g_2, r_2), f(g_3, r_3)] \cong [f(g_4, r_4), f(g_5, r_5)]$ holds if $[(g_2, r_2), (g_3, r_3)] \cong [(g_4, r_4), (g_5, r_5)]$.
Because f is a bijection, there exists one, and only one, element $(\gamma, \rho) \in G \times R$ such that $f(\gamma, \rho) = (f_1(\gamma, \rho), f_2(\gamma, \rho)) = (0_{G'}, 0_R) \in G' \times R$. Then, we can define an application $\delta : G \to G'$ such that $\delta(g) = f_1(g + \gamma, \rho)$.

Let us first demonstrate that δ is a morphism between ordered sets.

Let $g_0 \le g_1$ on G, then $(g_0, r) \le (g_1, r)$ on $G \times R$, for any $r \in R$. It follows that $f(g_0, r) \le f(g_1, r)$, that is, $f_1(g_0, r) < f_1(g_1, r)$, or $f_1(g_0, r) = f_1(g_1, r)$ and $f_2(g_0, r) \le f_2(g_1, r)$. In any case, $f_1(g_0, r) \le f_1(g_1, r)$, that is, f_1 preserves the order with respect to the first component (keeping constant the second component). In particular, if we take $r = \rho$, then we obtain $\delta(g_0) = f_1(g_0 + \gamma, \rho) \le f_1(g_1 + \gamma, \rho) = \delta(g_1)$. That is, δ preserves the order as well.

Note that f_1 also preserves the order with respect to the second component (keeping constant the first component). In fact, if $r_0 \le r_1$, then $(g, r_0) \le (g, r_1)$ for any $g \in G$. It follows that $f(g, r_0) \le f(g, r_1)$, that is, in any case, $f_1(g, r_0) \le f_1(g, r_1)$.

Let us now prove that δ is a morphism between the groups G and G'. Let g, g' be any two elements in G. Then, it holds that $[(\gamma, \rho), (g + \gamma, \rho)] \cong [(g' + \gamma, \rho), (g + g' + \gamma, \rho)]$ on $G \times R$. In fact, $g + \gamma - \gamma = g + g' + \gamma - (g' + \gamma)$ because G is an abelian group, and $\rho - \rho = \rho - \rho = 0_R$.

By assumption regarding f, it holds that $[f(\gamma, \rho), f(g + \gamma, \rho)] \cong [f(g' + \gamma, \rho), f(g + g' + \gamma, \rho)]$, thus implying $f_1(g + \gamma, \rho) - f_1(\gamma, \rho) = f_1(g + g' + \gamma, \rho) - f_1(g' + \gamma, \rho)$. It follows that $\delta(g + g') = f_1(g + g' + \gamma, \rho) = f_1(g + \gamma, \rho) - f_1(h, \rho) + f(g' + \gamma, \rho) = f_1(g + \gamma, \rho) - 0_{G'} + f(g' + \gamma, \rho) = \delta(g) + \delta(g')$. This concludes the proof that δ is a morphism between the groups G and G'.

Last, let us demonstrate that δ is a bijection. Let g_0, g_1 be any two distinct elements on G and assume $g_0 \le g_1$, then $(g_0 + \gamma, \rho) \le (g_1 + \gamma, \rho)$ on $G \times R$. By the congruence $[(\gamma, \rho), (g_0 + \gamma, \rho)] \cong [(g_1 + \gamma, \rho), (g_1 - g_0 + \gamma, \rho)]$ it holds that $f_1(g_0 + \gamma, \rho) - f_1(\gamma, \rho) = f_1(g_1 - g_0 + \gamma, \rho) - f_1(g_1 + \gamma, \rho)$, that is, $f_1(g_1 - g_0 + \gamma, \rho) + f_1(g_0 + \gamma, \rho) = f_1(g_1 + \gamma, \rho)$.

If $f_1(g_1 - g_0 + \gamma, \rho) = 0$, then it follows that $f_1(g + \gamma, \rho) = 0$ for any $g \in G$, because g_0, g_1 are arbitrary elements. Further, by considering $s' \in R$ such that $\rho \le s'$ it follows that:

(a) $0 = f_1(g + \gamma, \rho) \le f_1(g + \gamma, s') \le f_1(g + g' + \gamma, \rho) = 0$, that is, $f_1(g + \gamma, s') = 0$ for any s' such that $\rho \le s'$;

(b) by $[(g + \gamma, \rho), (g + \gamma, s')] \cong [(g + \gamma, 2\rho - s'), (g + \gamma, \rho)]$ it holds that $f_1(g + \gamma, s') - f_1(g + \gamma, \rho) = f_1(g + \gamma, \rho) - f_1(g + \gamma, 2\rho - s')$, i.e. $f_1(g + \gamma, 2\rho - s') = 0$ that implies, together with a), $f_1(g + \gamma, s') = 0$ for any $s' \in R$. A contradiction, because f is a bijection.

Then, $f_1(g_1 - g_0 + \gamma, \rho) \ne 0$, and thus $f_1(g_0 + \gamma, \rho) \ne f_1(g_1 + \gamma, \rho)$, implying $\delta(g_0) \ne \delta(g_1)$. In other words, δ is a one-to-one application.

Now we must demonstrate that δ is surjective.

Let $g_0' \in G'$, $r_0' \in R$, then there exists an element $(g_0, r_0) \in G \times R$ such that $f(g_0 + \gamma, r_0) = (g_0', r_0')$ because f is a bijection. Because f_1 preserves the order and is injective, for any $g_1, g_2 \in G$, such that $g_1 < g_0 < g_2$, it follows that $f_1(g_1 + \gamma, r) < f_1(g_0 + \gamma, r) = g_0' < f_1(g_2 + \gamma, r)$, that is, by changing the first component, the image of f_1 becomes different from g_0'. Then, it holds that $\{g_0'\} \times R \subseteq \{f(g_0 + \gamma, r) \mid r \in R\} = f(g_0 + \gamma, R)$, that is, the pairs on $G' \times R$, with g_0' as first component, can be obtained, through f, mapping from $G \times R$ only the pairs with $g_0 + \gamma$ as first component. Let us check if the reverse holds, that is, if $\Lambda = \{f(g_0 + \gamma, r) \mid r \in R\} \subseteq \{g_0'\} \times R$.

Let us assume, to achieve this goal, that there exists $r_1 \in R$ such that $f_1(g_0 + \gamma, r_1) = g_0'' \ne g_0'$ (we will arrive at a contradiction).

There are two possibilities: $r_1 < r_0$, that implies $g_0'' < g_0'$, or $r_0 < r_1$, that implies $g_0' < g_0''$.

Consider the first case. Because f_1 preserves the order with respect to the second component, it holds that $\{f_1(g_0 + \gamma, r) \mid r \in [r_1, r_0]\} \subseteq [g_0'', g_0']$. Note also, because f is bijective, that there are infinite values $r \in R$ such that $f_1(g_0 + \gamma, r) = g_0'$, more precisely, $|\Lambda| = |\{r \in R \mid f(g_0 + \gamma, r) = g_0'\}| = |f^{-1}(g_0', R)| = |R|$.

Among these values, there cannot exist any value r' such that $f_1(g_0 + \gamma, r') \ne g_0'$ because, otherwise, if $r_0 < r' < r_0'$, with $f_1(g_0 + \gamma, r_0) = f_1(g_0 + \gamma, r_0') = g_0'$, it would follow that $g_0' = f_1(g_0 + \gamma, r_0) < f_1(g_0 + \gamma, r') < f_1(g_0 + \gamma, r_0') = g_0'$, the latter not holding.

Let $r_2 = \inf\{r \in R \mid f(g_0 + \gamma, r) = g_0'\}$ that, as hypothesized, must exist and satisfy $r_1 < r_2 \le r_0$. We have two cases:

i) $r_2 \ne r_0$. Then, $f_1(g_0 + \gamma, r) = g_0'$ for any $r \in (r_2, r_0]$, and $f_1(g_0 + \gamma, r) < g_0'$ for any $r \in [r_1, r_2)$. Thus, we can define $y = \min\{(r_2 - r_1)/3, (r_0 - r_2)/3\}$ and consider the intervals $[(g_0 + \gamma, r_2 - y), (g_0 + \gamma, r_2 + y)]$, $[(g_0 + \gamma, r_2 + y), (g_0 + \gamma, r_2 + 3y)]$. As one may prove, such intervals are congruent. Then, their corresponding intervals through f must also be congruent. That is, at least for the first component, it holds that $f_1(g_0 + \gamma, r_2 + y) - f_1(g_0 + \gamma, r_2 - y) = f_1(g_0 + \gamma, r_2 + 3y) - f_1(g_0 + \gamma, r_2 + y)$, or $g_0' > f_1(g_0 + \gamma, r_2 - y) = g_0' - g_0' + g_0' = g_0'$. A contradiction;

ii) $r_2 = r_0$. Then, there exists, by the infiniteness of Λ, an element $r_3 > r_0$ such that $f_1(g_0 + \gamma, r_3) = g_0'$. Note that $f_1(g_0 + \gamma, r) = g_0'$ for any $r \in [r_0, r_3]$, and $f_1(g_0 + \gamma, r) < g_0'$ for any $r \in [r_1, r_0)$. Thus, we can define $y = \min\{(r_0 - r_1)/3, (r_3 - r_0)/3\}$ and can consider the intervals $[(g_0 + \gamma, r_0 - y), (g_0 + \gamma, r_0 + y)]$, $[(g_0 + \gamma, r_0 + y), (g_0 + \gamma, r_0 + 3y)]$. Such intervals are congruent as are their correspondences through f. This implies that, for the first component, $f_1(g_0 + \gamma, r_0 + y) - f_1(g_0 + \gamma, r_0 - y) = f_1(g_0 + \gamma, r_0 + 3y) - f_1(g_0 + \gamma, r_0 + y)$ or $g_0' > f_1(g_0 + \gamma, r_2 - y) = g_0' - g_0' + g_0' = g_0'$. A contradiction, again.

Thus, r_1 cannot be less than r_0.

In the second case, that is, $r_0 < r_1$, following an analogous process, $\{f_1(g_0 + \gamma, r) \mid r \in [r_0, r_1]\} \subseteq [g_0', g_0'']$ and let $r_2 = \sup\{r \in R \mid f(g_0 + \gamma, r) = g_0'\}$ that, as hypothesized, must exist and satisfy $r_0 \leq r_2 < r_1$. We have two cases:

i) $r_2 \neq r_0$. Then, $f_1(g_0 + \gamma, r) = g_0'$ for any $r \in [r_0, r_2)$, and $f_1(g_0 + \gamma, r) > g_0'$ for any $r \in (r_2, r_1]$. Let y, indeed, be the $\min\{(r_2 - r_0)/3, (r_1 - r_2)/3\}$ and consider the intervals $[(g_0 + \gamma, r_2 - 3y), (g_0 + \gamma, r_2 - y)]$, $[(g_0 + \gamma, r_2 - y), (g_0 + \gamma, r_2 + y)]$ that are still congruent. Then it holds that $f_1(g_0 + \gamma, r_2 - y) - f_1(g_0 + \gamma, r_2 - 3y) = f_1(g_0 + \gamma, r_2 + y) - f_1(g_0 + \gamma, r_2 - y)$, or $g_0' < f_1(g_0 + \gamma, r_2 + y) = g_0' - g_0' + g_0' = g_0'$. A contradiction;

ii) $r_2 = r_0$. Then, there exists, by the infiniteness of Λ, an element $r_3 < r_0$ such that $f_1(g_0 + \gamma, r_3) = g_0'$. Note that $f_1(g_0 + \gamma, r) = g_0'$ for any $r \in [r_3, r_0]$ and $f_1(g_0 + \gamma, r) > g_0'$ for any $r \in (r_0, r_1)$. Thus, we can define $y = \min\{(r_0 - r_3)/3, (r_1 - r_0)/3\}$ and can consider the intervals $[(g_0 + \gamma, r_0 - 3y), (g_0 + \gamma, r_0 - y)]$, $[(g_0 + \gamma, r_0 - y), (g_0 + \gamma, r_0 + y)]$. Such intervals are congruent as are their correspondences through f, implying $f_1(g_0 + \gamma, r_0 - y) - f_1(g_0 + \gamma, r_0 - 3y) = f_1(g_0 + \gamma, r_0 + y) - f_1(g_0 + \gamma, r_0 - y)$, or $g_0' < f_1(g_0 + \gamma, r_0 + y) = g_0' - g_0' + g_0' = g_0'$. A contradiction, again.

Thus, r_1 cannot be greater than r_0. This fact, together with the previous result implies $r_1 = r_0$, which is the final contradiction. Therefore, such an r_1 cannot exist.

It follows that $\{g_0'\} \times R = \{f(g_0 + \gamma, r) \mid r \in R\}$. In particular, setting $r = \rho$, it holds that $\delta(g_0) = f_1(g_0 + \gamma, \rho) = g_0'$. In other words, δ is an onto application.

4.- Conclusions.

To clarify the meaning of the second theorem, we may keep in mind that, as we noted at the end of the fourth step of Section 2, the validity of some axioms of *L* do not imply the validity of the same property on *T*, when we apply the quotient application.

Examples of non-isomorphic non-archimedean lines are defined on the following Cartesian product: $Z \times R$, $Q \times R$, $R \times R$, $(Z \times R) \times R$, $(Q \times R) \times R$, $(R \times R) \times R$, $(Z \times Q \times R) \times R$, ecc.

Bibliography

[1] Peano G. (1889), I Principii di Geometria logicamente esposti, da *Opere scelte* (1958), a cura dell'UMI, Cremonese, vol. 11, 59-78.

[2] Peano G. (1894), Sui Fondamenti della Geometria, tratto da *Opere scelte* (1959), a cura dell'UMI, Cremonese, vol. III, 115-157.

[3] Pasch M. (1882), *Vorlesungen fiber neuere Geometrie*, Lipsia.

[4] Morin U., Busulini F. (1966), *Elementi di Geometria per le Scuole Medie Superiori*, Quarta edizione, Cedam, Padova.

[5] Hilbert D. (1970), *Fondamenti della Geometria*, traduzione di Pietro Canetta dall'opera *Grundlagen der Geometrie* (1889), Feltrinelli, Milano.

[6] Manara C. F. (1991), Giuseppe Peano ed i Fondamenti della Geometria, in: Atti del Convegno "Peano e i Fondamenti della Matematica , Mucchi, Modena, 171-184.

[7] Marchi M. (1991), *L'Opera di Peano a la moderna Geometria di Incidenza*, in: Atti del Convegno "Peano e i Fondamenti della Matematica ", Mucchi, Modena, 197-212.
[8] Freguglia P. (1977), *Osservazioni inerenti alla Geometria sulla retta di G. Peano*, Archimede, 2, 95-103.
[9] Eugeni F., Furneri S., Mercanti F. (1999), *Una presentazione delle Geometrie non Archimedee*, in: Atti del Congresso Nazionale della Mathesis 1999, Edigrafital, Teramo, 101-111.
[10] Eugeni F., Mascella R., Un'assiomatica per la retta euclidea reale alla maniera di Peano, in: *Critica dei fondamenti* (a cura di F. Eugeni),, Edigrafital, Teramo, 2002, 37-62.
[11] Eugeni F., Mascella R., La retta euclidea reale a partire da una relazione d'ordine, *Periodico di Matematiche*, 3 (2001), 45-56.
[12] Eugeni F., Mascella R., Su alcuni modelli geometrici non archimedei, in: *Critica dei fondamenti* (a cura di F. Eugeni),, Edigrafital, Teramo, 2002, 63-90.
[13] Veronese G., *Sul postulato di continuità*, Rend. Reale Acc. Lincei, vol. VI, II sem., (1897), 161-167.
[14] Veronese G., *Segmenti e numeri trasfiniti*, Rend. Reale Acc. Lincei, vol. VII, II sem., (1898), 162-169.
[15] Veronese G., *La geometria non archimedea. Una questione di priorità*, Rend. Reale Acc. Lincei, vol. XIV, I sem., (1905), 347-351.
[16] Veronese G., *La geometria non archimedea*, Atti del IV Congresso intern. dei matematici, vol. I (1908), 197-208.